NIMEQ: MHD Equilibrium Solver for NIMROD

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Abstract

- A Grad-Shafranov equilibrium solver is developed within the NIMROD framework to create plasma profiles for realistic geometry. The traditional Grad-Shafranov operator is converted to a pure divergence allowing the use of standard regularity conditions for the quantity $\psi/R^2$ in simply connected domains. The resulting equation is solved in the weak form using a finite element representation. A Picard scheme is used to advance the nonlinear iteration.
Outline

• Introduction
• Outline of solver
  – Derivation of finite element Grad-Shafranov operator
  – Inhomogeneous boundary conditions
  – Nonlinear iteration
  – Calculation of equilibrium fields
  – NIMROD Implementation
• Verification of solver accuracy
  – Constant lambda cylindrical pinch
  – Circular cross section tokamak
  – SSPX geometry
• Conclusions & future work
INTRODUCTION: The NIMROD code is a computational laboratory designed to study 3D plasmas for fusion based applications [Phys. Plasmas 10 (2003) 1727].

- NIMROD uses a spectral element expansion in 2 dimensions and finite Fourier series in the third.
  - Spectral element expansion provides NIMROD with the flexibility to study plasma in complex geometries.
    - High-order accuracy resolves extreme anisotropy.
  - Fourier expansion in third dimension requires a degree of symmetry.
    - Linear periodic or toroidal geometry
  - Allows timesteps that are large relative to normal-mode propagation times.
NIMROD computations use Grad-Shafranov equilibria in many applications.

• The capability to create simple 1D equilibria exists within NIMROD’s preprocessor.

• For more complex geometries, an equilibrium is created from an external source and then interpolated to a NIMROD mesh.
  – The process of interpolation introduces numerical errors into the equilibrium.
  – Software designed to reconstruct equilibria from experimental measurements may sacrifice accuracy for speed.
  – The propagation of these errors though the simulation increases as the disparity of spatial and temporal scales in the physics increase.

• NIMROD simulations use equilibria as initial states or as fixed data, where the code just evolves perturbations.
Simulations used to study linear instabilities in SSPX discharges illustrate the practicality of a native equilibrium solver.

- **Cylindrical pinch**
  - SSPX profiles have been approximated with linear cylindrical pinches.
  - Analytic solutions can be used to generate equilibrium.
  - Simplifications result in loss of geometric effects.
- **Realistic mesh representing entire flux conserver**
  - Geometric effects from gun and wall are included.
  - Initial conditions can be read from reconstructed equilibria.
  - A native solve can be used to scan profiles in this realistic geometry.
Creating an equilibrium solver within NIMROD allows the full use of spectral element flexibility and accuracy.

- Using the same expansion to create equilibria and run MHD simulations eliminates all interpolation errors.
  - Best possible equilibrium for a particular grid.
- NIMROD users will be able to modify the solver to meet demands of specific applications.
  - Magnetosphere plasma profiles
  - Parameterized tokamaks, RFPs, and spheromaks
  - Refine equilibria from other codes
- Modularity of NIMROD provides most of the computational tools needed for the equilibrium solver.
Outline of solver: The Grad-Shafranov equation describes 2 dimensional axisymmetric plasma equilibria with no flow.

\[ \Delta^* \psi = -F(\psi)F'(\psi) - \mu_0 R^2 p'(\psi) \]

\[ \Delta^* = \frac{\partial^2}{\partial R^2} - \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial z^2} \]

- Nonlinear, elliptic, partial differential equation for the flux function \( \psi(R,z) \)
  - \( F \) and \( p \) are independent functions of \( \psi \) and need to be specified.
  - \( F = RB_\phi \quad p=\text{pressure} \)
- In a linear geometry \( (\phi=>z) \), the del star operator reduces to the Laplacian in \( (x,y) \).

\[ \nabla^2 \psi = -F(\psi)F'(\psi) - \mu_0 p'(\psi) \]
The Grad-Shafranov operator can be expressed as a total divergence.

\[ \vec{A} \equiv R^2 \nabla \left( R^{-2} \psi \right) \quad \Xi = R^{-2} \psi \]

\[ \nabla \cdot \vec{A} = \frac{\partial^2}{\partial R^2} \psi - \frac{1}{R} \frac{\partial}{\partial R} \psi + \frac{\partial^2}{\partial z^2} \psi = \Delta^* \psi \]

- Expanding \( \psi \) shows that \( \Xi \) is well behaved near \( R=0 \) as long as \( \psi_0=0 \).

\[ \psi = \psi_0 + \psi_2 R^2 + \ldots \]

\[ \Xi = \frac{\psi_0}{R^2} + \psi_2 + \ldots \]

- We choose \( \psi_0=0 \) for the arbitrary constant so that \( \Xi \) satisfies standard regularity conditions.
A finite element algebraic equation is formulated for the dependent variable \( \Xi \).

\[
\nabla \cdot (R^2 \nabla \Xi) = G(\Xi, R), \quad G(\Xi, R) \equiv -\mu_0 FF' - \mu_0 p'R^2
\]

\[
\int \left( \nabla \cdot (R^2 \nabla \Xi) \right) \alpha_i(R, z) dV = \int G \alpha_i dV \quad \text{Weak form of Grad-Shafranov equation}
\]

\[
\Xi = \sum \Xi_j \alpha_j(R, z) \quad \text{Expand dependent variable with finite element basis functions}
\]

\[
\sum \int R^2 \left( \frac{\partial \alpha_i}{\partial R} \frac{\partial \alpha_j}{\partial R} + \frac{\partial \alpha_i}{\partial Z} \frac{\partial \alpha_j}{\partial Z} \right) dV \Xi_j = -\int G \alpha_i dV + \text{surfacet TERMS}
\]

\[
\sum M_{ij}^* \Xi_j = g_i \quad \text{No contribution for Dirchlet boundary conditions on } \Xi
\]
A surface integral is performed to calculate \( \psi \) along the boundary.

- The change in the flux function along the boundary is given by
  \[
  \Delta \psi = - \int_{x_1}^{x_2} R(\vec{B} \cdot \hat{n}) dl
  \]

- Normal magnetic field is prescribed.
- Gaussian quadrature is used to perform surface integral.
- The value \( \Xi \) along the border is used as essential conditions on the expansion.
- In toroidal geometry regularity places further constraints on \( \psi \).

\[
\lim_{r \to 0} \frac{B_z}{2} \to \psi_2
\]
A modified Picard scheme is used to perform nonlinear iteration.

\[ \Xi^{n+1} = w \, M^{-1} \, G^n + (1 - w) \Xi^n \]

- \( w \) is a relaxation parameter that provides stabilization to nonlinear iteration.
- Convergence is checked by comparing the residual of the nonlinear solve with the magnitude of

\[ \frac{\| \Delta^* \psi + F F' + \mu_0 R^2 P' \|}{\| \Delta^* \psi \|} \equiv \text{error} \]

- In practice, Picard scheme is adequate.
  - Computation takes 1-2 minutes on laptops and workstations.
- A Newton iteration can be added to accelerate convergence near solution.
The equilibrium $\vec{B}(R,z)$, $\vec{J}(R,z)$, and $p(R,z)$ are calculated from $\Xi$, $F(\psi)$, and $p(\psi)$.

\[
R \vec{B} = R \nabla \phi \times \nabla \psi + F \hat{e}_\phi
\]
\[
\mu_0 \vec{J} = F' \nabla \psi \times \nabla \phi + \Delta^* \psi \nabla \phi
\]

- Chain rule is applied to calculate $\vec{B}$ from the known quantity $\Xi$ and then $\vec{J}_{pol}$ is calculated from $B_{pol}$.
  - Nimrod uses $RB_\phi$ and $R^{-1}J_\phi$ in the representation of equilibrium fields.

\[
R \vec{B}_{pol} = \hat{e}_\phi \times \left( R^2 \nabla \Xi + 2 R \Xi \hat{e}_r \right)
\]
\[
RB_\phi = F
\]
\[
\mu_0 \vec{J}_{pol} = - \vec{B}_{pol} F'
\]
\[
R^{-1}J_\phi = - FF' / \mu_0 R^2 - P'
\]
NIMEQ flowchart shows the steps in the nonlinear iteration.

1. Read mesh and magnetic field on boundary from input file.
2. Generate finite element Grad-Shafranov operator.
3. Perform surface integration along the boundary to calculate inhomogeneous contribution.
4. Make an initial guess for nonlinear iteration.
5. Calculate $FF'$ and $P'$ for the current flux distribution.
6. Calculate the residual error and check for convergence.
7. Solve algebraic system for next value of $\Xi$.
8. Calculate right hand side from $FF'$ and $p'$.
9. Check to see if the number of iterations exceeds the maximum allowed value.
10. Generate input file for NIMROD.

If the number of iterations exceeds the maximum allowed value, go back to step 4. If the residual error is within the convergence criteria, go to step 10. Otherwise, go back to step 6.
Several basic models for $F$ and $p$ have been implemented.

- A normalized ring flux $Y$ has been calculated to distinguish between physics in open and closed flux regions.
  - $Y=0$ on the magnetic axis
  - $Y=1$ on separatrix and open flux region

- Three models for $F$
  - $F=f_{\text{open}}+f_1(1-Y)+4f_2Y(Y-1)$
  - $F=f_{\text{open}}+(f_{\text{axis}}-f_{\text{open}})(1-3Y^2+2Y^3)$
  - $F=f_0+f_1\psi$

- Two models for pressure
  - $P=P_0$
  - $P=P_{\text{open}}+(P_{\text{axis}}-P_{\text{open}})(1-3Y^2+2Y^3)$
Verification of solver accuracy: Analytic solutions for the magnetic field in a constant lambda cylindrical pinch are used to benchmark solver.

- Magnetic fields are functions radius only.
  - \( B_\varphi = -B_0 J_1(\lambda R) \)
  - \( B_z = B_0 J_0(\lambda R) \)
  - \( B_r = 0 \)
  - Nimrod uses \((R,z,\varphi)\) coordinates instead of \((R,\theta,z)\) coordinates.
  - \( B_z \) reverses sign at \( \lambda R = 2.404 \).

- Three different computation grids are used to test different aspects of the solver.
  - Rectangular grid in a toroidal configuration with B specified along the entire boundary.
  - Rectangular grid in a toroidal configuration with periodic boundary conditions in the \( z \)-direction and the flux specified on the outer boundary.
  - Circular grid with periodicity in the \( z \)-direction and the flux specified on the outer boundary.
Plots of equilibrium fields generated with a rectangular mesh show agreement with analytic solutions.
Contours of $B_R$ for rectangular grids with different boundary conditions display the surface integrator accuracy.

- Values of $\vec{B}$ specified on boundary
- 16x16 elements with 4$^\text{th}$ order polynomials
- Max $B_R = 1.863 \times 10^{-10}$
- $B_0 = 1.0$

- Periodic boundary conditions in $z$ with $\psi$ specified on outer surface
- 16x16 elements with 4$^\text{th}$ order polynomials
- Max $B_R = 1.833 \times 10^{-12}$
- $B_0 = 1.0$
Comparison of equilibrium fields on circular grid verifies the solver in non-toroidal systems.

- Fields for $\lambda = 1$ solutions agree with theoretical values.
- Plots of $\vec{B}_{pol}$ show that $B_R$ is small and depends only on $\psi$.
- Nonlinear iterations fail to converge for equilibria with $\lambda = 2.5$ (axial field reversal).
  - Additional work is being done to improve convergence.
Axisymmetric nonlinear simulations of a circular cross-section tokamak test quality of equilibrium with finite pressure.

Generated equilibrium is used as initial condition for nonlinear simulation.
  - Major Radius =2 Minor radius =1
  - Cubic pressure profile with $\beta \approx 0.3$
  - 18x18 circular grid with 5th order polynomials

Simulation ran for 7500 $\tau_A$.

A change in peak pressure of 0.046% is observed.
An equilibrium for SSPX is generated on a grid that represents its flux conserver.

- Quadratic model for $F$ is used with $RB_{\phi i}$ on the magnetic axis set to 0.17 Tm.
- Cubic model for pressure with $u_0P$ on axis set to 0.1
  - $\beta = 2/3$
- $\tau_A = 1.88 \times 10^{-7}$s
- 20x28 mesh with 4th order polynomial elements
- Equilibrium was designed with no open flux surfaces to aid simulations of nonlinear evolution.
Axisymmetric nonlinear simulations of SSPX equilibrium test the quality of equilibrium.

- Simulation is run for $\sim 100 \ t_A$
- Peak pressure changed by $\sim 0.16\%$
- An isotropic number density diffusivity of 100 m$^2$/s is used for numerical stability.
Conclusions & Future Work

• A Grad-Shafranov solver has been developed for the NIMROD code.
• Fields created for a constant lambda cylindrical pinch show good agreement with analytical solution.
• Nonlinear simulations of equilibria ran over numerous Alfvén times suggest that the plasma is in equilibrium.
• Addition benchmarking is being performed.
• Functionality to create grid and generate initial magnetic field along the boundary will be created within NIMEQ.
• Use equilibria from NIMEQ to study SSPX and other configurations.